Do Stokes' and Gauss's theorems hold in curved spacetime?

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The happy students of Prof. Risto Tammelo - the former head of Theoretical Physics in our University have learned the following formula:

$$\oint_S \vec{A}\,\vec{\sigma} = \int\int\int \mathrm{div}\,\vec{A}\;dV$$

in the flat Euclidean Universe with metric $ds^2 = dx^2 + dy^2 + dz^2$. Hereby $dV = dx\,dy\,dz$, the $\vec{\sigma}$ is the vectorized surface element of the closed surface S. The integration on the right is throughout the inner of this surface S. It is the Gaussian Theorem. In the index notation it looks like

$$\oint A^i\,\sigma_i = \int\int\int A^k{}_{,k}\,dV\,.$$

Here and after the index ", u" means a partial derivative, but index "; u" means the co-variant derivative (the one, which uses Christoffel Symbols $\Gamma^{\nu}{}_{\mu\,\alpha}$. They are used, if there is a curviness in the problem). It is derived from the definition of the divergence: $\mathrm{div}\,A = A^k{}_{,k}$. Note, what it is not derivable from $A^k{}_{,k} = 0$. Here the k takes all available values in this universe and the expression is scalar, ie. invariant and physical. Therefore, it is expected, what in curved coordinates of flat spacetime, or in curved spacetime holds following law

$$\oint A^{\nu}\,\sigma_{\nu} = \int\int\int A^{\mu}{}_{;\mu}\,\sqrt{-g}\,d\Omega\,,$$

which is also scalar. Here $\sigma_{\nu} = n_{\nu}\,\sqrt{-g}\,d\sigma$, where the surface element $d\sigma = dx^i\,dx^j\,dx^k$ with $i \neq j \neq k$ and $i, j, k \in (t, x, y, z)$, the coordinate volume $d\Omega = dt\,dx\,dy\,dz$. Here the x, y, z are not necessarily the Cartesian coordinates, ie. in some cases might be $x \equiv r$, $y \equiv \theta$, $z \equiv \phi$. All this is possible, because the vector basis of the "curvature coordinates" can be only orthonormal.

Here must hold $|n^{\nu}\,n_{\nu}| = 1$, and n_{ν} must be orthogonal to the surface $\varphi(t, x, y, z) = const$ of integration: $n_{\nu} = \varphi_{,\nu}$. Note, what (e.g. in example $A^{\mu\,\nu} = C^{\mu}\,U^{\nu}$, where C^{μ} are constants) in curved situation the formula does not hold for second (and higher) rank tensors. Perhaps even the flat spacetime suffers this problem in a curved coordinate systems.

Now the theory is over, let us apply our knowledge!

I. APPLICATION

The continuity equation $J^{\nu}{}_{,\nu} = 0$, where the 4-current of fluid or dust is $J^{\nu} = \rho\,u^{\nu}$. In curved spacetime it is an absurd: what is conserved, if there is no consistent theory of conservation??? But now it describes the rest-mass conservation. The latter means, that the number of particles in fluid (or dust) remains the same.

We can get insides on the closed Friedman Universe with help of this simple tricks:

If you have the Scalar Curvature R, then $R_{;\nu} \equiv A_\nu$ is needed vector. If you have Scalar $S = R^{\mu\nu} R_{\mu\nu}$ then needed vector is $S_{;\nu}$.

If you have tensor $R^{\mu\nu}$ then needed vector is $K^\nu := R^{\nu\mu} \delta^t{}_\mu$, where $\delta^t{}_\mu = (1, 0, 0, 0)$ are coordinates of the temporal basis vector of the curvature coordinates.

Keep in mind, what in presence of Dark Energy is not holding the old good formula $G^{\mu\nu} = 8\pi T^{\mu\nu}$, and in presence of Dark Matter even such promising formula $G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi T^{\mu\nu}$ fails. Why, you're asking? The Dark Matter and Dark Energy have no experimental reason to be included in the Matter section of General Relativity: they are not in $T^{\mu\nu}$. See also vixra.org/abs/1512.0347

II. APPLICATION FOR FRIEDMANN UNIVERSE

. Take the simple metric $ds^2 = -dt^2 + a^2(t)\,(dr^2/(1 - k\,r^2) + r^2 d\Omega^2)$, where $k = -1, 0, 1$. Let the A^ν does not depend on space coordinates, only on time. The $\sqrt{-g} \sim a^3(t)$, therefore holds (after the taken the t-derivative)

$$(A^t\, a^3)_{,\,t} = A^\nu{}_{;\,\nu}\, a^3\,.$$

It is really amazing, what all the choices I made ($A_\nu = R_{;\nu}$, $A_\nu = (R^{\alpha\beta} R_{\alpha\beta})_{;\nu}$, $A_\nu = (R^{tt})_{;\nu}$)) do satisfy the latter equation identically and without restriction of the $a(t)$: it holds with any vector A^ν and with any scale function $a(t)$ and for any kind of Universe (ie, the constant k is also arbitrary).

If you apply the formulas for the energy-momentum tensor of the perfect fluid $T^\mu{}_\nu = (-\rho(t),\, p(t),\, p(t),\, p(t))$ then it turned out to be

$$\rho = \rho_0\,(a_0/a(t))^3\,.$$

Because must hold the continuity equation $(\rho\, u^\nu)_{;\,\nu} = 0$ with $u^\nu = (1, 0, 0, 0)$. This case coincides perfectly with experimental data: see vixra.org/abs/1304.0086.