Problem 1.4 At time $t = 0$ a particle is represented by the wave function

$$\Psi(x,0) = \begin{cases} A\dfrac{x}{a}, & \text{if } 0 \le x \le a, \\ A\dfrac{(b-x)}{(b-a)}, & \text{if } a \le x \le b, \\ 0, & \text{otherwise,} \end{cases}$$

where A, a, and b are constants.

(a) Normalize Ψ (that is, find A, in terms of a and b).

(b) Sketch $\Psi(x,0)$, as a function of x.

(c) Where is the particle most likely to be found, at $t = 0$?

(d) What is the probability of finding the particle to the left of a? Check your result in the limiting cases $b = a$ and $b = 2a$.

(e) What is the expectation value of x?

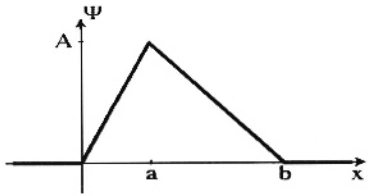
Problem 1.4

(a)

$$1 = \frac{|A|^2}{a^2}\int_0^a x^2 dx + \frac{|A|^2}{(b-a)^2}\int_a^b (b-x)^2 dx = |A|^2 \left\{ \frac{1}{a^2}\left(\frac{x^3}{3}\right)\Big|_0^a + \frac{1}{(b-a)^2}\left(-\frac{(b-x)^3}{3}\right)\Big|_a^b \right\}$$

$$= |A|^2\left[\frac{a}{3} + \frac{b-a}{3}\right] = |A|^2\frac{b}{3} \Rightarrow \boxed{A = \sqrt{\frac{3}{b}}.}$$

(b)



(c) At $\boxed{x = a.}$

(d)

$$P = \int_0^a |\Psi|^2 dx = \frac{|A|^2}{a^2}\int_0^a x^2 dx = |A|^2\frac{a}{3} = \boxed{\frac{a}{b}.}\begin{cases} P = 1 \text{ if } b = a, \checkmark \\ P = 1/2 \text{ if } b = 2a. \checkmark \end{cases}$$

(e)

$$\langle x\rangle = \int x|\Psi|^2 dx = |A|^2\left\{\frac{1}{a^2}\int_0^a x^3 dx + \frac{1}{(b-a)^2}\int_a^b x(b-x)^2 dx\right\}$$

$$= \frac{3}{b}\left\{\frac{1}{a^2}\left(\frac{x^4}{4}\right)\Big|_0^a + \frac{1}{(b-a)^2}\left(b^2\frac{x^2}{2} - 2b\frac{x^3}{3} + \frac{x^4}{4}\right)\Big|_a^b\right\}$$

$$= \frac{3}{4b(b-a)^2}\left[a^2(b-a)^2 + 2b^4 - 8b^4/3 + b^4 - 2a^2b^2 + 8a^3b/3 - a^4\right]$$

$$= \frac{3}{4b(b-a)^2}\left(\frac{b^4}{3} - a^2b^2 + \frac{2}{3}a^3b\right) = \frac{1}{4(b-a)^2}(b^3 - 3a^2b + 2a^3) = \boxed{\frac{2a+b}{4}.}$$

$$\langle x \rangle = \int x|\psi|^2 dx = |A|^2 \left\{ \frac{1}{a^2}\int_0^a x^3 dx + \frac{1}{(b-a)^2}\int_a^b x(b-x)^2 dx \right.$$

$$= \frac{3}{b}\left\{ \frac{1}{a^2}\frac{x^4}{4}\Big|_0^a + \frac{1}{(b-a)^2}\int_a^b x(b^2 - 2bx + x^2)\, dx \right\}$$

$$= \frac{3}{b}\left\{ \frac{a^2}{4} + \frac{1}{(b-a)^2}\int_a^b (b^2 x - 2bx^2 + x^3)\, dx \right\}$$

$$= \frac{3}{b}\left\{ \frac{a^2}{4} + \frac{1}{(b-a)^2}\left(b^2\frac{x^2}{2} - 2b\frac{x^3}{3} + \frac{x^4}{4} \right)\Big|_a^b \right\}$$

$$\langle x \rangle = \frac{3}{b}\left\{ \frac{a^2}{4} + \frac{1}{(b-a)^2}\left(\frac{b^4}{2} - \frac{2b^4}{3} + \frac{b^4}{4} - b^2\frac{a^2}{2} + 2b\frac{a^3}{3} - \frac{a^4}{4} \right) \right\}$$

$$\frac{b^4}{2} - \frac{2b^4}{3} + \frac{b^4}{4} = \frac{5}{12} b^4$$

$$\langle x \rangle = \frac{3}{4}\left\{ \frac{a^2}{4} + \frac{1}{(b-a)^2}\left(\frac{5}{12} b^4 - \frac{b^2 a^2}{2} + \frac{2ba^3}{3} + \frac{a^4}{4} \right) \right\}$$